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ON CATENARY CURVE AS VARIATIONAL PROBLEM WITH CONSTRAINTS AND IT IS APPLICATIONS

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ABSTRACT

In this paper we used the calculus of variation method to prove that the corresponding curve of a heavy a chain with length C or cable supported at its end which hang freely in a uniform gravitational field is catenary curve, some applications of the catenary curve are also present.

Keywords: *Maximum and minimum. Eular-Lagrange equation, canenary curve.*

I. INTRODUCTION

Calculus of variations which beginnings in (1696) with John Bernoulli is a branch of mathematics dealing with the optimization of physical quantities. Our objective is to drive the equation of catenary curve under some constrains. The word catenary is derived from the Latin word catena, which means "chain". The English word catenary, in (1638), Galileo says that a hanging cord is an approximate parabola, and he correctly observes that this approximation improves as the curvature gets smaller and is almost exact when the elevation is less than 45°see[1] That the curve followed by a chain is not a parabola was proven by Joachim Jungius (1587–1657); this result was published posthumously in 1669. The application of the catenary to the construction of arches is attributed to Robert Hooke, whose "true mathematical and mechanical form" in the context of the rebuilding of St Paul's Cathedral alluded to a catenary Some much older arches approximate catenaries, an example of which is the Arch of Taq-i Kisra in Ctesiphon see [2]

In (1671) Hooke announced to the Royal Society that he had solved the problem of the optimal shape of an arch, and in 1675 published an encrypted solution as a Latin anagram in an appendix to his description of Helioscopes see [3] where he wrote that he had found "a true mathematical and mechanical form of all manner of Arches for Building." He did not publish the solution to this anagram in his lifetime, but in 1705 his executor provided it as Ut pendet continuum flexile, sic stabit contiguum rigidum inversum meaning "As hangs a flexible cable so, inverted, stand the touching pieces of an arch."

In 1691 Gottfried Leibniz, Christiaan Huygens, and Johann Bernoulli derived the equation in response to a challenge by Jakob Bernoulli. David Gregory wrote a treatise on the catenary in 1697.

Euler proved in (1744) that the catenary is the curve which, when rotated about the x-axis, gives the surface of minimum surface area (the catenoid) for the given bounding circles. Nicolas Fuss gave equations describing the equilibrium of a chain under any force in 1796.

II. DEFINITION1: FUNCTIONAL

The quantity z is called a functional of $f(x)$ in the interval [a, b] if it depends on all the values of $f(x)$in [a, b].

Euler–Lagrange equation: The Euler–Lagrange equation was developed in the 1750s by Euler and Lagrange in connection with their studies of the tautochrone problem. This is the problem of determining a curve on which a weighted particle will fall to a fixed point in a fixed amount of time, independent of the starting point.

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Lagrange solved this problem in 1755 and sent the solution to Euler. Both further developed Lagrange's method and applied it to mechanics, which led to the formulation of Lagrangian mechanics. Their correspondence ultimately led to the calculus of variations, a term coined by Euler himself in 1766.

Mathematical formulation of Lagrange equation: The Euler–Lagrange equation is an equation satisfied by a function, q, of a real argument, t, which is a stationary point of the functional

$$S(q) = \int_a^b L\big(t, q(t), q'(t)\big)\, dt$$

where: q is the function to be found:

$$q: [a,b] \subset R \to X, t \to x = q(t)$$

such that q is differentiable, $q(a) = x_a$, and $q(b) = x_b$; q' is the derivative of q

$$q': [a,b] \to T_{q(t)}X, t \to v = q'(t)$$

$T\,X$ being the tangent bundle of X defined by

$$T\,X = \bigcup_{x \in X} \{x\} \times T_x\,X$$

L is a real-valued function with continuous first partial derivatives:

$$L: [a,b] \times T\,X \to R, (t,x,v) \to L(t,x,v)$$

The Euler–Lagrange equation, then, is given by:

$$L_x\big(t, q(t), q'(t)\big) - \frac{d}{dx} L_v\big(t, q(t), q'(t)\big) = 0$$

Where L_x and L_v denote the partial derivatives of L with respect to the second and third arguments, respectively.
If the dimension of the space X is greater than 1, this is a system of differential equations, one for each component:

$$\frac{\partial}{\partial q} L\big(t, q(t), q'(t)\big) - \frac{d}{dt}\left(\frac{\partial}{\partial q'} L\big(t, q(t), q'(t)\big)\right) = 0$$

III. VARIATIONAL WITH CONSTRAIN

Maximum and minimum problem often involve certain constraints, for instance an extremum of the function $f = f(x,y)$ may be thought under the restriction that x and y are be related to the equation $G = G(x,y) = C$ where C is constant.

Theorem: The problem of stationary values of $f = f(x,y)$ subject to constrain $G = G(x,y) = C$ is equivalent to the problem of stationary values without constraints of the function

$$H(x,y) = F(x,y) + \lambda G(x,y)$$

For some constants λ provided at least one of the partial derivatives $\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$
Does not vanishes at critical point.

Euler-Lagrange equation for function with constraints
The Euler-Lagrange equation for function $H(x,y) = F(x,y) + \lambda\, G(x,y)$ is

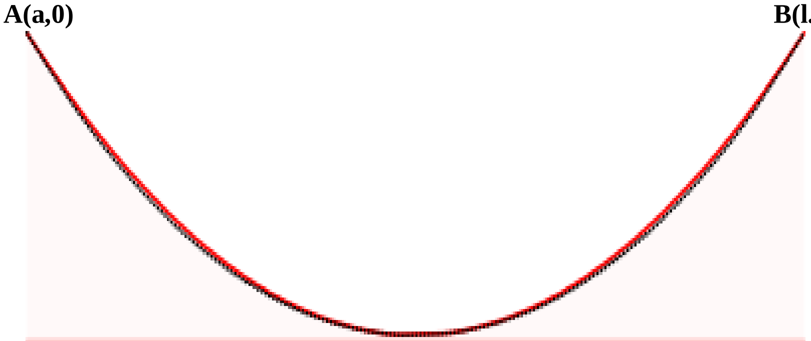
$$\frac{\partial H(x,y)}{\partial y} - \frac{d}{dx}\left[\frac{\partial H(x,y)}{\partial y'}\right] = 0$$

$$\left(\frac{\partial F(x,y)}{\partial y} - \frac{d}{dx}\left[\frac{\partial F(x,y)}{\partial y'}\right]\right) + \lambda\left(\frac{\partial G(x,y)}{\partial y} - \frac{d}{dx}\left[\frac{\partial G(x,y)}{\partial y'}\right]\right) = 0 \quad \ldots\ldots\ldots\ldots (2)$$

Catenary curve: The catenary curve is the curve that described by a uniform chain hanging from two supports in a uniform gravitational field.

Geometrical properties:
Over any horizontal interval, the ratio of the area under the catenary to its length equals a, independent of the interval selected. The catenary is the only plane curve other than a horizontal line with this property. Also, the geometric centroid of the area under a stretch of catenary is the midpoint of the perpendicular segment connecting the centroid of the curve itself and the x-axis see[6]

Statement of the problem:
A heavy chain with length C when it hung between two fixed points $A(\, a, 0)$ and $B\ (l, b)$ find the equilibrium shape. (see the fig)



In mechanics the equilibrium shape will be when the potential energy is minimum , in sense of calculus of variation we have to minimize the functional

$$J = \rho g \int_0^L y ds = \rho g \int_0^L y\sqrt{1 + (y')^2}\, dx$$

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Where ρds is the mass of the element of arc ds
The constrain is

$$N = \int_0^L ds = \rho g \int_0^L \sqrt{1+(y')^2}\, dx = c$$

Let

$$H(x,y,y') = y\sqrt{1+(y')^2} + \lambda\sqrt{1+(y')^2} = (y+\lambda)\sqrt{1+(y')^2}$$

The factor ρds may be dropped for our purpose. Apply Eular – Lagrange (2)

$$\frac{\partial}{\partial y}\left((y+\lambda)\sqrt{1+(y')^2}\right) - \frac{d}{dx}\left[\frac{\partial}{\partial y'}\left((y+\lambda)\sqrt{1+(y')^2}\right)\right] = 0$$

$$= \sqrt{1+(y')^2} - \frac{d}{dx}\left((y+\lambda)\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$= \sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}} - \frac{y''(y+\lambda)}{(1+(y')^2)^{\frac{3}{2}}} = 0$$

After simplify we obtain

$$\frac{1}{(y+\lambda)} = \frac{y''}{1+(y')^2} \qquad \dots\dots\dots\dots\dots\dots (3)$$

put $p = y'$, then (3) will be in the form:

$$\frac{p\, d\, p}{1+p^2} = \frac{dy}{y+\lambda}$$

this is O.D.E of p with solution

$$1+p^2 = \frac{(y+\lambda)^2}{k^2} \qquad \dots\dots\dots\dots\dots. (4)$$

where k is constant. Since $p = y'$ then (4) is

$$\frac{dy}{\sqrt{(y+\lambda)^2 - k^2}} = \frac{dx}{k}$$

Again this is O.D.E with solution so

$$\cosh^{-1}\left(\frac{y+\lambda}{k}\right) = \frac{x}{k} + h$$

or

$$y+\lambda = k\ \cosh\left(\frac{x}{k}+h\right) \qquad \dots\dots\dots\dots\dots\dots\dots (5)$$

Where h is the second constant of integration, the three constants $\lambda, k,$ and h are determined from the conditions.
Firstly since

$$y(l) = b, \qquad y(0) = a$$

then

$$b+\lambda = k\ \cosh\left(\frac{l}{k}+h\right) \qquad \dots\dots\dots\dots\dots\dots\dots (6)$$

$$a+\lambda = k\ \cosh(h) \qquad \dots\dots\dots\dots\dots\dots\dots (5)$$

$$\int_0^l \sqrt{1+(y')^2} = C \qquad \dots\dots\dots\dots\dots\dots\dots.. (7)$$

From (5)

$$y' = \sin sh\left(\frac{x}{k}+h\right) \qquad \dots\dots\dots\dots\dots\dots\dots.. (8)$$

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IV. APPLICATIONS

Catenary curve has many applications in different areas of science and real life such as:

A charge in a uniform electric field moves along a catenary (which tends to a parabola if the charge velocity is much less than the speed of light c see [7]. The surface of revolution with fixed radii at either end that has minimum surface area is a catenary revolved about the x-axis see [8].

A Freely-hanging electric power cables (especially those used on electrified railways) can also form a catenary

A Catenary arches are often used in the construction of kilns. To create the desired curve, the shape of a hanging chain of the desired dimensions is transferred to a form which is then used as a guide for the placement of bricks or other building material

A Catenary bridges

In free-hanging chains, the force exerted is uniform with respect to length of the chain, and so the chain follows the catenary curve. The same is true of a simple suspension bridge or "catenary bridge," where the roadway follows the cable see [9],[10]

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simple suspension bridge rare essentially thickened cables, and follow a catenary curve. A stressed ribbon bridge is a more sophisticated structure with the same catenary shape see [11]

(Stressed ribbon bridges, like this one in Maldonado, Uruguay, also follow a catenary curve, with cables embedded in a rigid deck.)

Also we find that the silk on a spider's web forming multiple elastic catenaries.

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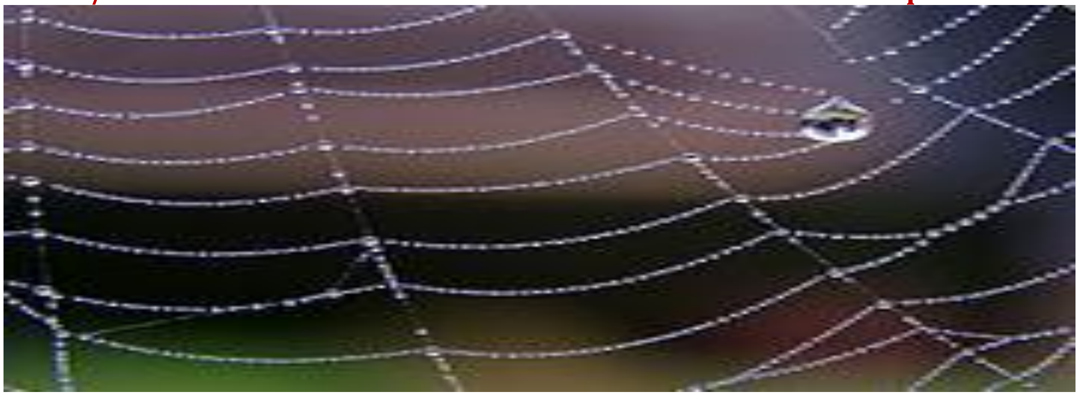

V. CONCLUSION

In physics and geometry, a **catenary**[p] is the **curve** that an idealized hanging chain or cable assumes under its own weight when supported only at its ends. The **curve** has a U-like shape, superficially similar in appearance to a parabola, but it is not a parabola: it is a (scaled, rotated) graph of the hyperbolic cosine.

The catenary curve is applicable to many everyday problems such as kiln construction and minimization of surface areas, and is even seen in the St Louis Arch. A di_eren-tial equation modeling a hanging chain of either uniform or variable density will procure the catenary curve. This is important in its applications for architecture and other problems.

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